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Static Solutions for Brane Models with a Bulk Scalar Field

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Abstract

We present static solutions of the 5-dimensional Einstein equations in the brane-world scenario by using two different approaches for the stabilization of the extra dimension. Assuming a "phenomenological" stabilization mechanism, that creates a non-vanishing $\hat{T}^5_5$ in the bulk, we construct a two-brane model, which allows both branes to have positive self-energies. We then consider a candidate theory for the dynamical stabilization, through the introduction of a massless scalar field in the bulk, which interacts with the branes. We find exact static solutions for the metric and scalar field in the bulk and demonstrate that the inter-brane distance is determined by the parameters of scalar field-brane interactions. However, these solutions are always accompanied by a correlation between the bulk cosmological constant, the brane self-energies and the interaction terms of the scalar field with the branes and thus cannot be considered as candidates for the phenomenologically viable stabilized geometry. We find that the aforementioned correlation cannot be avoided even in the case of a single-brane solution with positive self-energy where the fifth dimension ends on a singularity.

1 Introduction

Two of the most serious problems which confronts unified theories today are the hierarchy problem and the cosmological constant. While supersymmetry can stabilize the hierarchy [1], the necessity to input mass scales which differ by many orders of magnitude persists. In some respects, the cosmological constant is even more severe, as many potential contributions to the vacuum energy density must cancel to extremely high precision. It is quite plausible that the solution to both of these problems lies beyond 4-dimensional field theory. Indeed there have been several recent attempts at attacking both of these problems in the context of higher dimensional theories in the case of the hierarchy problem [2, 3, 4, 5] and in the case of the cosmological constant [6, 7].

In theories with extra dimensions (large or small), phenomenology and cosmology must be restricted to a 3-brane solution in the larger theory. Indeed, considerable attention has been focused on one and two 3-brane solutions. In particular, in the static solution of Randall and Sundrum [2], the scale factor a, is derived to be exponentially decreasing as one moves away from a 3-brane with positive tension. In a space-time with a compact extra dimension, a negative tension is necessary, and a mass hierarchy can be established between the two branes.

In the absence of a stabilization mechanism for the modulus of the extra dimension (radion), non-static solutions appeared problematic as the cosmological expansion rate was found to depend on the energy density ($H \simeq \rho$) rather than its square root as in the standard FRW Universe [8, 9, 10]. Solutions to this problem by adding both matter and a cosmological constant on the two branes inevitably led to the wrong sign for gravity on one of the branes [11, 12, 13, 14].

It was subsequently realized that the "normal" form of the Friedmann equation is intimately related with the stabilization of the extra dimension [15, 16, 17, 18]. Ideally, this should be accomplished by a mechanism which works without any fine tuning of the "input" parameters and can be universally applied for any equation of state on the brane. A consequence of such a stabilization is the existence of (55)-component of the energy-momentum tensor in the bulk, proportional to the trace of the energy-momentum tensor on the brane [15]. It was further shown [17] that this component arises due to the shift of the minimum of the radion potential in response to the presence of the brane. This way, the relation $T^5_5 \simeq -(2R)^{-1}(\rho - 3p)$ arises naturally and is independent of the details of the stabilization. See also Ref. [19] for related constraints.

The apparent simplicity of the static solution for the metric in the RS model is based on the exact fine-tuning of the bulk and brane cosmological constants. The fine tuning is exacerbated when perturbations of the brane self-energies with matter densities are included. In this case, fine tuning between the energy density and pressure components on the two

different branes is needed. This issue was readdressed in Ref. [16], where a phenomenological stabilization potential for the transverse scale factor was introduced. The potential removes the need for the correlation between matter densities on different branes.

Given the necessity for a radion-fixing potential in any realistic generalization, it is now fair to question the necessity of the negative energy brane. Indeed, it should be possible to construct a solution for two positive self-energy branes if the distance between two branes is stabilized. It was shown in [17, 20] that the general solution to the Einstein's equations for the 3-space scale factor in the presence of the negative bulk cosmological constant admits *cosh*-like behaviour. For this solution, the usual 4D Friedmann equations for matter trapped on a single brane can be easily obtained. However, the cancellation of the effective cosmological constant on the brane is an extra fine-tuning condition. Because of the minimum in a, the same *cosh*-like solutions should be able to accommodate two positive self-energy branes placed on opposite sides of the minimum. Here we plan to study such static two-brane configurations with positive self-energies. We will determine the allowed values of the parameters in this model and investigate the possible hierarchy between scale factors on two different branes.

Irrespective of the size of the extra dimension, it is natural to expect that the brane self-energy is large, on the order of the fifth power of the fundamental 5-dimensional Plank scale. On the other hand, the matter density, ρ, is small in these units no matter how low the fundamental scale might be. This is true even in the extreme case when $M_5 \sim 1$ TeV, $\rho \ll \text{TeV}^4$. As such, it is clear that a natural mechanism for the cancellation of the effective cosmological constant on the brane is another very important question which has to be resolved in order to connect the brane-world proposal to reality. To this end, we first study static solutions to Einstein equations and neglect the matter density ρ. The time independence of these solutions automatically means that the effective cosmological constant on the brane is equal to zero. If such solutions are found, one can then perturb them by including a small ρ in order to get a consistent phenomenological and cosmological description.

The stabilization of the extra dimension with a bulk scalar field was discussed by Goldberger and Wise in Refs. [21]. See also Ref. [22]. There, the original RS solution was modified by including a scalar field in the bulk, which has an interaction (potential) with the two branes. This stabilization does not evade the fine tuning, which in Goldberger-Wise approach is the same fine tuning as in Refs. [2, 23], that is the fine tuning between brane self-energies and bulk cosmological constant. It is important, however, to study this mechanism in more detail in order to understand to what extent it depends on the specific assumptions concerning the scalar field, its potential in the bulk and interactions with the branes.

The purpose of this letter is two-fold. First, we derive static solutions to Einstein's

equations with two branes with positive self-energies by allowing the value of T_{55} to be non-zero in the bulk. We find that such a solution can accommodate any positive values of the brane self-energies between zero and a limiting value corresponding to the brane self-energy in the Randall-Sundrum model. The ratio of the scale factors on the two branes is determined through the deviations of the brane self-energies from this limiting value. Secondly, we find exact static solutions to the Einstein's equations in the presence of a massless scalar field, with the bulk energy-momentum tensor given *only* by a cosmological constant and the energy-momentum tensor of this field. We argue that in this case the proper stabilization of the extra dimension and/or cancellation of the effective cosmological constant on the brane is not possible unless some specific fine tuning conditions are satisfied. Finally, we present the single-brane configuration with the spacetime ending on a true singularity in the extra dimension and comment on the subject of fine tuning in this case. This solution generalizes the Randall-Sundrum model and shows that the exponentially decaying scale factor eventually ends on the singularity, situated at the point in the extra dimension determined through the strength of the brane-scalar field interaction.

2 Static two-brane models with phenomenological stabilization of extra dimension

We start with the description of the geometrical framework of our analysis. The line-element of the 5-dimensional spacetime is given by the following ansatz

$$ds^2 = a^2(y)\,(-dt^2 + \delta_{ij}dx^i dx^j) + b^2(y)dy^2\,, \tag{2.1}$$

where $\{t, x^i\}$ and y denote the usual, 4-dimensional spacetime and the extra dimension, respectively. Here, we focus only on static configurations of the spacetime background and ignore any time dependence of the conformal factor a and the scale factor b along the extra dimension. Without loss of generality, we can assume $b = 1$.

We will also assume that the two 3-branes with positive self-energies Λ_1 and Λ_2 are located at $y = y_1$ and $y = -y_2$, respectively. In the region between the two 3-branes, a non-vanishing cosmological constant Λ_B is assumed to exist. The action functional that describes the above (4+1)-dimensional, gravitational theory has the following form

$$S = -\int d^4x\, dy\, \sqrt{-\hat{g}}\left\{\frac{M_5^3}{16\pi}\,\hat{R} + \Lambda_B + \Lambda_1\,\delta(y-y_1) + \Lambda_2\,\delta(y+y_2)\right\}. \tag{2.2}$$

In the above, M_5 is the fundamental 5-dimensional Planck mass and the hat denotes 5-dimensional quantities. The existence of some stabilization mechanism is also assumed which ensures that the distance between the two branes remains fixed. According to Refs. [15, 16, 17, 18], this requires a bulk value for $\hat{T}^5_5$ different from $-\Lambda_B$. In this sense, the

solutions that we derive here, are generalizations of the Randall-Sundrum constructions which allow the existence of a non-trivial bulk value of $\hat{T}^5_5$ and, consequently, positive self-energies for both branes.

The variation of the action (2.2) with respect to the 5-dimensional metric tensor $\hat{g}_{MN}$ leads to Einstein's equations, which for the spacetime background (2.1) take the form

$$\hat{G}_{00} = -3a^2\left\{\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right\} = \hat{\kappa}^2\,\hat{T}_{00}\,, \tag{2.3}$$

$$\hat{G}_{ii} = 3a^2\left\{\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right\} = \hat{\kappa}^2\,\hat{T}_{ii}\,, \tag{2.4}$$

$$\hat{G}_{55} = 6\left(\frac{a'}{a}\right)^2 = \hat{\kappa}^2\,\hat{T}_{55}\,, \tag{2.5}$$

where $\hat{\kappa}^2 = 8\pi G_N^{(5)} = 8\pi/M_5^3$ and the primes denote differentiation with respect to y. Note that the (05)-component of Einstein's equations vanishes identically due to the time-independence of the line-element (2.1).

Taking into account the contributions from the bulk cosmological constant and the brane self-energies, the energy-momentum tensor that appears on the rhs of Einstein's equations can be written as

$$\hat{T}^M{}_N = \Big[\Lambda_B + \Lambda_1\,\delta(y-y_1) + \Lambda_2\,\delta(y+y_2)\Big]\,(-\delta^M{}_N)\,. \tag{2.6}$$

In addition, we allow the (55)-component to deviate from this form due to the existence of radius stabilization potential [17]. It is straightforward to see that, for the above choice, eqs. (2.3) and (2.4) reduce to the same differential equation for the conformal factor $a(y)$. In the bulk, this can be conveniently rewritten as

$$(a^2)'' = \frac{2\hat{\kappa}^2}{3}(-\Lambda_B)\,a^2\,. \tag{2.7}$$

In the case of a negative bulk cosmological constant, $\Lambda_B < 0$, the general solution for $a^2(y)$ in the bulk is given by an arbitrary linear combination of rising and falling exponents,

$$a^2(y) = A\exp\left(-\sqrt{\frac{2\hat{\kappa}^2}{3}|\Lambda_B|}y\right) + B\exp\left(\sqrt{\frac{2\hat{\kappa}^2}{3}|\Lambda_B|}y\right)\,, \tag{2.8}$$

where A and B are integration constants. The conformal factor must also satisfy boundary conditions at $y = y_1$ and $y = -y_2$ which depend on the brane self-energies, Λ_1 and Λ_2. It is clear that $\Lambda_{1,2} > 0$ can be accommodated only if the solution for $a^2(y)$ in the bulk, is not monotonic. In this case, $a^2(y)$ is given by a hyperbolic cosine, and without a loss of

generality we can place the minimum of this function, y_0, at the point $y = 0$ and redefine y_1 and $-y_2$ accordingly. Then, the solution for the conformal factor takes the form

$$a^2(y) = a_0^2 \cosh\left(\sqrt{\frac{2\hat{\kappa}^2}{3}|\Lambda_B|y}\right) . \tag{2.9}$$

The embedding of the two 3-branes with zero thickness in the 5-dimensional manifold creates a discontinuity of the first derivative, with respect to y, of the conformal factor $a(y)$. This, in turn, leads to the appearance of a Dirac delta function in the (00) and (ii)-components of Einstein's equations (2.3)-(2.4), where its second derivative appears. By matching the coefficients of the delta functions in the aforementioned equations, the following *jump* conditions at the points $y = y_1$ and $y = -y_2$ emerge

$$\frac{[a']_1}{a_1} = -\frac{\hat{\kappa}^2}{3}\Lambda_1 , \qquad \frac{[a']_2}{a_2} = -\frac{\hat{\kappa}^2}{3}\Lambda_2 , \tag{2.10}$$

where the subscripts 1 and 2 denote quantities evaluated at $y = y_1$ and $y = -y_2$, respectively. Using the expression (2.9), the above conditions can be rewritten as

$$\tanh\left(\sqrt{\frac{2\hat{\kappa}^2}{3}|\Lambda_B|\, y_1}\right) = \frac{\Lambda_1}{\sqrt{6|\Lambda_B|/\hat{\kappa}^2}} \equiv \frac{\Lambda_1}{\Lambda_{RS}} , \tag{2.11}$$

$$\tanh\left(\sqrt{\frac{2\hat{\kappa}^2}{3}|\Lambda_B|\, y_2}\right) = \frac{\Lambda_2}{\sqrt{6|\Lambda_B|/\hat{\kappa}^2}} = \frac{\Lambda_2}{\Lambda_{RS}} . \tag{2.12}$$

As one can see, the position of the branes is determined by their self-energies. Moreover, these two conditions show that the static solution (2.9) can arise only if

$$0 \leq \Lambda_1, \Lambda_2 \leq \Lambda_{RS} . \tag{2.13}$$

The limiting case of $\Lambda_i = \Lambda_{RS}$ corresponds to $y_i \to \infty$ and effectively reproduces the solution of Ref. [23] with the exponentially decaying conformal factor. The ratio of scale factors on the two branes can be expressed in terms of "detuning" of Λ_i from the limiting values Λ_{RS},

$$\frac{a_2^2}{a_1^2} = \sqrt{\frac{\Lambda_{RS}^2 - \Lambda_1^2}{\Lambda_{RS}^2 - \Lambda_2^2}} . \tag{2.14}$$

In principle, this ratio can be very large or very small, depending on the relative size of these detunings. In order to solve the hierarchy problem, we must assume that the observable matter fields are localized to the brane with the smaller scale factor. Thus, we have demonstrated that the gauge hierarchy problem can be resolved by a "geometrical" explanation à la Ref. [2] with two positive self-energy branes.

Clearly, the above solution cannot arise without a contribution to the (55)-component of the energy-momentum tensor, other than $-\Lambda_B$. The value of $\hat{T}_{55}$ consistent with the solution (2.9) can be easily determined by substituting the solution for the conformal factor in eq. (2.5), and is found to be

$$\hat{T}_{55} = |\Lambda_B| - \frac{|\Lambda_B|}{\cosh^2\left(\sqrt{\frac{2\hat{\kappa}^2}{3}|\Lambda_B|}\, y\right)} \,. \tag{2.15}$$

If the inter-brane distance, $y_1 + y_2$, is large as compared to the length scale given by $1/\sqrt{\frac{2\hat{\kappa}^2}{3}|\Lambda_B|}$, the (55)-component of the energy-momentum tensor deviates from $|\Lambda_B|$ only in the vicinity of $y = 0$ near the minimum of the scale factor. Using eqs. (2.11) and (2.12), we can rewrite this expression in the following form,

$$\hat{T}_{55} = |\Lambda_B| \left(1 - \frac{a_i^4}{a^4}\left[1 - \frac{\Lambda_i^2}{\Lambda_{RS}^2}\right]\right) = |\Lambda_B| - \frac{a_i^4}{a^4}\frac{\Lambda_i^2\hat{\kappa}^2}{6\sinh^2\left(\sqrt{\frac{2\hat{\kappa}^2}{3}|\Lambda_B|}\, y_i\right)}, \tag{2.16}$$

which coincides with the expression obtained in Ref. [17]. a_i, Λ_i, and y_i can be evaluated on *either* brane.

We should stress at this point that the distance between the two branes (or, equivalently, the volume of the extra dimension) turns out to be fixed in terms of the brane self-energies and the bulk cosmological constant. In the limit of small bulk cosmological constant, the relations (2.11) and (2.12) take a remarkably simple form and can be combined to give the result

$$\Lambda_1 + \Lambda_2 + 2(y_1 + y_2)\Lambda_B = 0. \tag{2.17}$$

This is nothing other than the condition of mutual cancellation between bulk and brane contributions to the effective cosmological constant, and, as such, is an extra fine-tuning condition which the radius stabilization has to satisfy. When we treat this stabilization "phenomenologically", by introducing a (55)-component for the energy-momentum tensor in the bulk, the mechanisms which could ensure this cancellation, simply cannot be addressed. Thus, we proceed to the considerations of 5D gravity plus a scalar field in the bulk interacting differently with the two branes, which was proposed in [21] to be a viable *dynamical* stabilization mechanism.

3 Gravity and a massless scalar in extra dimensions

In this section, we assume the existence of a scalar field, $\hat{\phi}$, in the bulk, in addition to a bulk cosmological constant Λ_B. We, now, choose the two 3-branes to be located at the points $y = 0$ and $y = L$. The bulk scalar field is minimally coupled to gravity but may

have different interactions with the two branes. The action functional of the theory, now, takes the form

$$S = -\int d^4x\, dy\, \sqrt{-\hat{g}}\,\Bigg\{\frac{M_5^3}{16\pi}\,\hat{R} + \Lambda_B + \frac{1}{2}\,\partial_M\hat{\phi}\,\partial^M\hat{\phi} + V_B(\hat{\phi}) \\ + \Big[\Lambda_1 + V_1(\hat{\phi})\Big]\,\delta(y) + \Big[\Lambda_2 + V_2(\hat{\phi})\Big]\,\delta(y-L)\Bigg\}\,. \qquad (3.18)$$

In the above expression, V_B, V_1, and V_2 are the bulk potential and the brane interactions of the scalar field on the brane 1 and 2, respectively. As before, Λ_1 and Λ_2 are the constant self-energies of the two branes. Non-vanishing bulk potentials were also considered in [24].

In the presence of the bulk scalar field, the Einstein's equations (2.3)-(2.5) are supplemented by the equation of motion for the scalar field, which has the form

$$\frac{1}{a^4}\,\frac{d}{dy}\left(a^4\,\frac{d\hat{\phi}}{dy}\right) = \frac{\partial V_B(\hat{\phi})}{\partial\hat{\phi}} + \frac{\partial V_1(\hat{\phi})}{\partial\hat{\phi}}\,\delta(y) + \frac{\partial V_2(\hat{\phi})}{\partial\hat{\phi}}\,\delta(y-L)\,. \qquad (3.19)$$

The energy-momentum tensor of the theory is also modified compared to the expression (2.6) of the previous section. The interaction terms of the scalar field on the two branes, V_1 and V_2, will contribute to the total brane self-energies while the bulk energy-momentum tensor may now be written as

$$\hat{T}^M_{\ \ N} = -\Lambda_B\,\delta^M_{\ \ N} + \hat{T}^M_{\ \ N}(\hat{\phi})\,, \qquad (3.20)$$

where

$$\hat{T}_{MN}(\hat{\phi}) = \partial_M\hat{\phi}\,\partial_N\hat{\phi} - \hat{g}_{MN}\left[\frac{1}{2}\,\partial_P\hat{\phi}\,\partial^P\hat{\phi} + V_B(\hat{\phi})\right]. \qquad (3.21)$$

Let us, first, concentrate on the equation of motion of the scalar field in the bulk where the last two terms on the rhs of eq. (3.19) vanish. In order to understand to which extent the proposed stabilization of the extra dimension with the scalar field [21] depends on the specific assumptions about its self-interaction, we take the potential in the bulk to be identically zero. We also notice that the choice $V_B(\hat{\phi}) = 0$ allows us to easily integrate the lhs of eq. (3.19) with respect to y and find $\hat{\phi}'$ in terms of the conformal factor $a(y)$. This, in turn, will lead to the determination of the conformal factor in the presence of the scalar field in the bulk, i.e. the backreaction of the scalar field on the spacetime geometry. Integrating eq. (3.19), we obtain the result

$$\hat{\phi}'(y) = \frac{c\,a_0^4}{a^4(y)}\,, \qquad (3.22)$$

where c is a constant and $a_0 = a(y=0)$. When the above expression is combined with Einstein's equations (2.3)-(2.5) and the expression for the energy-momentum tensor in the

bulk (3.20), we are led to the following system of differential equations for $a(y)$

$$\frac{a''}{a} + \left(\frac{a'}{a}\right)^2 = \frac{\hat{\kappa}^2}{3}\left(-\Lambda_B - \frac{c^2 a_0^8}{2a^8}\right), \tag{3.23}$$

$$2\left(\frac{a'}{a}\right)^2 = \frac{\hat{\kappa}^2}{3}\left(-\Lambda_B + \frac{c^2 a_0^8}{2a^8}\right). \tag{3.24}$$

Rearranging the above two equations, we are led to a single differential equation

$$\frac{(a^4)''}{4a^4} = -\frac{2\hat{\kappa}^2}{3}\,\Lambda_B\,, \tag{3.25}$$

which can be easily integrated to give the solution for the conformal factor $a(y)$ in terms of the bulk cosmological constant. The substitution of the solution in any of the original equations (3.23)-(3.24) and the boundary condition $a(y = 0) \equiv a_0$ will determine any arbitrary integration constants. In this way, we obtain the following solution

$$a^4(y) = a_0^4\,\frac{|y - y_0|}{y_0}\,, \qquad y_0 = \sqrt{\frac{3}{4\hat{\kappa}^2 c^2}}\,, \tag{3.26}$$

in the case of vanishing Λ_B, which is similar to the solution found in [7], and

$$a^4(y) = a_0^4\,\frac{\sin(\omega|y - y_0|)}{\sin(\omega y_0)}\,, \qquad y_0 = \frac{1}{\omega}\,Arc\sin\sqrt{\frac{2\Lambda_B}{c^2}}\,, \tag{3.27}$$

or

$$a^4(y) = a_0^4\,\frac{\sinh(\omega|y - y_0|)}{\sinh(\omega y_0)}\,, \qquad y_0 = \frac{1}{\omega}\,Arc\sinh\sqrt{\frac{2|\Lambda_B|}{c^2}}\,, \tag{3.28}$$

for positive or negative, respectively, Λ_B. The parameter ω appearing in the above expressions is defined as

$$\omega^2 = \frac{8\hat{\kappa}^2}{3}\,|\Lambda_B|\,. \tag{3.29}$$

Note that all the above solutions are characterized by the existence of a spacetime singularity at $y = y_0$, where the conformal factor vanishes while the first derivative of the scalar field (3.22) diverges. By placing a second brane at a point $y = L < y_0$, we can ensure that the above solutions are well defined everywhere. Hereafter, we concentrate on the case of a negative bulk cosmological constant, however, similar conclusions can be drawn in the other two cases as well.

The inhomogeneity in the distribution of matter in the 5-dimensional manifold leads to a discontinuity of the first derivative, with respect to y, not only of the conformal factor $a(y)$, but of the bulk scalar field $\hat{\phi}(y)$, too. By following the same method as in section 2, i.e. by matching the coefficients of the delta functions in the equations where their second

derivatives appear, the following *jump* conditions, for both the conformal factor and the scalar field, emerge

$$\frac{[a']_0}{a_0} = -\frac{\hat{\kappa}^2}{3}\left[\Lambda_1 + V_1(\hat{\phi}_0)\right], \qquad [\hat{\phi}']_0 = \left.\frac{\partial V_1(\hat{\phi})}{\partial \hat{\phi}}\right|_{y=0}, \tag{3.30}$$

$$\frac{[a']_L}{a_L} = -\frac{\hat{\kappa}^2}{3}\left[\Lambda_2 + V_2(\hat{\phi}_L)\right], \qquad [\hat{\phi}']_L = \left.\frac{\partial V_2(\hat{\phi})}{\partial \hat{\phi}}\right|_{y=L}, \tag{3.31}$$

where the subscripts 0 and L denote quantities evaluated at $y=0$ and $y=L$, respectively. In the above, we have used the fact that the energy-momentum tensor on the two branes is generated by the interaction terms of the bulk scalar field and the brane self-energies. By using the expressions (3.22) and (3.28), for the first derivative of the scalar field and the solution for the conformal factor in the bulk, respectively, the above conditions may be written as

$$\omega\,\coth(\omega y_0) = \frac{2\hat{\kappa}^2}{3}\left[\Lambda_1 + V_1(\hat{\phi}_0)\right], \qquad 2c = \left.\frac{\partial V_1(\hat{\phi})}{\partial \hat{\phi}}\right|_{y=0}, \tag{3.32}$$

$$\omega\,\coth\!\big[\omega\,(y_0 - L)\big] = -\frac{2\hat{\kappa}^2}{3}\left[\Lambda_2 + V_2(\hat{\phi}_L)\right], \qquad 2c\,\frac{a_0^4}{a_L^4} = -\left.\frac{\partial V_2(\hat{\phi})}{\partial \hat{\phi}}\right|_{y=L}. \tag{3.33}$$

A close examination of the above equations renders the allowed values for the brane self-energies, "dressed" with the interaction with $\hat{\phi}$. Assuming that the positive self-energy brane is situated at $y=0$, we arrive at the following allowed ranges for the effective self-energies:

$$\Lambda_{RS} \;\leq\; \Lambda_1 + V_1(\hat{\phi}_0) \leq \infty\,, \tag{3.34}$$

$$-\infty \;\leq\; \Lambda_2 + V_2(\hat{\phi}_L) \leq -\Lambda_{RS}\,, \tag{3.35}$$

from which we immediately conclude that this solution cannot accommodate two positive self-energy branes. Of course, we can choose both Λ_1 and Λ_2 to be positive and remain consistent with the boundary conditions (3.32)-(3.33) provided that the potential on one of the branes is negative making the "dressed" brane self-energy negative, i.e. $\Lambda_2 + V_2(\hat{\phi}_L) < 0$. The fact that one of the two branes has a negative total energy density follows from the form of the solution (3.28) for the conformal factor $a(y)$ in the bulk. This expression describes a monotonically decreasing function that interpolates between the two boundary values a_0 and a_L.

The remaining nontrivial condition which relates $\hat{\phi}_L$ to $\hat{\phi}_0$ is the continuity of the $\hat{\phi}$ field in the bulk,

$$\hat{\phi}_L = \hat{\phi}_0 + \int_0^L \hat{\phi}'(y)\,dy = \hat{\phi}_0 + c\int_0^L \frac{a_0^4}{a^4(y)}\,dy\,. \tag{3.36}$$

This equation, together with the boundary conditions and explicit forms for a^4 and $\hat{\phi}'$, lead to an *overdetermined* set of algebraic equations. This means that, in general, no static solution can be found unless one extra fine tuning on the original parameters, Λ_1, Λ_2, V_1 and V_2, is imposed. We note that the form of the interaction terms of the scalar field $\hat{\phi}$ on the two branes completely determines the ratio of the values of the conformal factor on the two branes. More specifically,

$$\left(\frac{a_0}{a_L}\right)^4 = \frac{\sinh(\omega y_0)}{\sinh[\omega\,(y_0 - L)]} = -\frac{(\partial_{\hat{\phi}} V_2)_{y=L}}{(\partial_{\hat{\phi}} V_1)_{y=0}}\,, \tag{3.37}$$

from which we further conclude that the derivatives of the interaction terms on the branes with respect to $\hat{\phi}$ should have opposite signs in order to achieve static solutions. The above relation also leads to the determination of the distance L between the two branes in terms of the fundamental parameters of the theory. In the limit of large $|\Lambda_B|$ and $\omega y_0, \omega\,(y_0 - L) \gg 1$, eq. (3.37) is simplified and leads to the result

$$L = \sqrt{\frac{3}{8\hat{\kappa}^2|\Lambda_B|}}\,\ln\left|\frac{(\partial_{\hat{\phi}} V_2)_{y=L}}{(\partial_{\hat{\phi}} V_1)_{y=0}}\right|, \tag{3.38}$$

which resembles the one derived by Goldberger and Wise [21]. In that case, the distance between the two branes remains fixed as long as the bulk scalar field assumes different vacuum expectation values on the two branes. As we can see from the example above, the distance between the branes is completely determined by the requirement of the time independence of the metric, equivalent to the cancellation of the effective cosmological constant. This conclusion is quite generic and holds for arbitrary interaction terms. Thus, in the case of the massless scalar, the fixed inter-brane distance is the consequence of the fine-tuning of cosmological constant rather than a true dynamical stabilization. Similarly, in the limit of small cosmological constant and small $\omega\, y_0$ and $\omega\,(y_0 - L)$, $a^4(y)$ becomes a linear function of y and the distance between the two branes is given by the expression

$$L = y_0\left(1 - \left|\frac{(\partial_{\hat{\phi}} V_1)_{y=0}}{(\partial_{\hat{\phi}} V_2)_{y=L}}\right|\right) = \sqrt{\frac{3}{\hat{\kappa}^2}}\left(\frac{1}{|(\partial_{\hat{\phi}} V_1)_{y=0}|} - \frac{1}{|(\partial_{\hat{\phi}} V_2)_{y=L}|}\right). \tag{3.39}$$

In the above, we have used the definitions (3.29) and the jump condition of the scalar field on the two branes. Once again, the distance between the two branes is uniquely determined and the derivatives of the interaction terms of the bulk scalar field on the branes should be different.

As an illuminating example, we consider the case of linear interaction terms, i.e. $V_1(\hat{\phi}) = \alpha\,\hat{\phi}$ and $V_2(\hat{\phi}) = \beta\,\hat{\phi}$. According to eq. (3.37), the coefficients α and β should be chosen in such a way as to satisfy $\alpha\,\beta < 0$. This statement is rather important since, unless the two branes have opposite "charges" with respect to ϕ, no static solutions arise in the above

framework. Then, the expression for the distance L between the two branes is simplified and is found to be

$$L = \sqrt{\frac{3}{8\hat{\kappa}^2|\Lambda_B|}}\, \ln\left|\frac{\beta}{\alpha}\right| , \tag{3.40}$$

in the case of a large bulk cosmological constant, while in the opposite case, we obtain

$$L = \sqrt{\frac{3}{\hat{\kappa}^2}}\left(\frac{1}{|\alpha|} - \frac{1}{|\beta|}\right) . \tag{3.41}$$

We now turn to the jump conditions that the solution for the conformal factor must satisfy on the two branes. Working again in the limit of large ωy_0 and $\omega\,(y_0 - L)$ and rearranging the jump conditions for $a(y)$ that appear in eqs. (3.32)-(3.33), we obtain the following conditions

$$\sqrt{\frac{6\,|\Lambda_B|}{\hat{\kappa}^2}} = -\Big[\Lambda_2 + V_2(\hat{\phi}_L)\Big] = \Lambda_1 + V_1(\hat{\phi}_0)\, . \tag{3.42}$$

The only remaining free parameter is the value of the scalar field on one of the two branes as $\hat{\phi}_L$ and $\hat{\phi}_0$ are related as follows

$$\hat{\phi}_L \simeq \hat{\phi}_0 + \frac{c}{\omega}\exp[-\omega(y_0 - L)]. \tag{3.43}$$

Similarly, in the limit of small ωy_0 and $\omega\,(y_0 - L)$ we obtain the relations

$$\frac{3}{2\hat{\kappa}^2} = -\Big[\Lambda_2 + V_2(\hat{\phi}_L)\Big]\,(y_0 - L) = \Big[\Lambda_1 + V_1(\hat{\phi}_0)\Big]\, y_0\, , \tag{3.44}$$

where the values of the scalar fields on the two branes are related by

$$\hat{\phi}_L = \hat{\phi}_0 + c y_0 \ln\frac{y_0}{y_0 - L}\, . \tag{3.45}$$

By choosing, for example, $\hat{\phi}_0$ to satisfy the condition $\Lambda_1 + V_1(\hat{\phi}_0) = \Lambda_{RS}$ in eq. (3.42), we are left with one fine tuning imposed on some combination of the fundamental parameters of the theory. The above result leads to the conclusion that, despite the presence of the bulk scalar field, the stabilization of the extra dimension still relies on the correlation that holds between the energy densities of the two branes and the bulk cosmological constant. In the limit V_1, $V_2 \to 0$, we recover the condition that holds between the bulk and brane cosmological constants in the case of the Randall-Sundrum model [2]. In that case, every distance L between the two branes is acceptable as long as the correlation between the energy densities of the two branes holds. In our case, for non-vanishing V_1 and V_2, a unique value of the distance L emerges which is mainly determined by the first derivatives of the interaction terms with respect to $\hat{\phi}$. However, once the interaction terms have been chosen,

the consistency of the solution, and thus the viability of the whole scenario, relies on the careful choice of the two self-energies, Λ_1 and Λ_2, in such a way as to satisfy the constraint (3.42). Alternatively, for fixed Λ_i, one must fine tune the parameters of the interaction terms V_i according to eq. (3.42) and this fine-tuning will change the distance between the two branes.

Another solution of the differential equation (3.25) given in terms of $cosh(\omega|y-y_0|)$ was rejected being inconsistent with the original equations (3.23) and (3.24). Such a solution, if acceptable, would describe a conformal factor characterized by the existence of a minimum at $y = y_0$ with both branches going upwards as one approaches the two branes. Mathematically, this solution would be consistent with the equations of motion only if the sign of the kinetic term of the bulk scalar field in eq. (3.18) were exactly the opposite. If we treat this case formally, both of the energy densities of the branes could be positive, however, the correlation between these two would still remain.The “wrong” sign of the kinetic term for the bulk field would correspond to a tachyonic mode and signal an intrinsic instability of such a construction. The appearance of tachyonic modes, in the absence of a monotonic configuration of the bulk scalar field along the extra dimension, was also pointed out in [22].

It appears that the only solution without a fine tuning between the brane self-energies and the bulk cosmological constant is the single-brane configuration with the extra dimension ending on a singularity. Indeed, going back to the solution for the conformal factor $a(y)$, eq. (3.28), and the first derivative of the bulk scalar field, eq. (3.19), we observe that the former quantity vanishes, while the latter diverges, at $y = y_0$. By evaluating the scalar curvature R, which is given by the expression

$$R = \omega^2 \left[\frac{3}{4} \coth^2(\omega|y - y_0|) - 2 \right], \tag{3.46}$$

one can easily check that a true spacetime singularity occurs at the point $y = y_0$. The solution for the conformal factor is still given by eq. (3.28) while the size of the extra dimension is set by the position of the singular point which can be found from the boundary conditions for $\hat{\phi}'$. In the case of a linear interaction of the bulk scalar field with the single brane, $V(\hat{\phi}) = \alpha\,\hat{\phi}$, the position of the singular point is given by

$$y_0 = \sqrt{\frac{3}{8\hat{\kappa}^2|\Lambda_B|}}\, Arc \sinh \frac{4\sqrt{|\Lambda_B|}}{|\alpha|}\,. \tag{3.47}$$

The boundary condition for the scale factor can be satisfied by the appropriate choice of $\hat{\phi}_0$. By performing an analysis similar to that of Ref. [4], we can easily see that the conservation of energy and momentum is not violated near the singularity for any massless particle, as well as for any massive excitations independent of y, propagating in the given spacetime background. In the limit of small cosmological constant in the bulk, $|\Lambda_B| \ll \alpha^2$, the distance to the singular point is inversely proportional to the size of the coupling α. In the opposite

limit of small coupling constant, $|\Lambda_B| \gg \alpha^2$, the solution for the scale factor is simply a falling exponent everywhere apart from the small ω^{-1} vicinity of y_0. Thus, in this limit, this solution is basically the one found by Randall and Sundrum [23] with the exponential tail being cut off at the finite distance y_0. For a vanishing value of the coupling, $\alpha \to 0$, this point is at infinity, $y_0 \to \infty$, the correlation between the brane self-energy and the bulk cosmological constant reappears and the solution coincides exactly with that in Ref. [23]. The presence of a singularity was recently advocated to solve the hierarchy problem [4] and the cosmological constant problem [7].

However, the absence of a second brane at $y < y_0$ does not mean that the system is not overdetermined. An accurate consideration of the singularity suggests that the consistency of the solution requires fixing boundary conditions for the conformal factor and the scalar field which is equivalent to assuming certain source terms at the singularity [25]. In order to have a consistent treatment of the boundary conditions at the singularity, it is helpful to return to our two-brane solution and consider the limit $\Lambda_2 \to -\infty$. The boundary condition for the scalar field requires that $|(\partial_{\hat{\phi}} V_2)_{y=L}| \to \infty$, as it can be easily seen from eq. (3.37). It can be further shown that the two limits have to be taken in a *correlated* way, in order to fulfill the condition (3.44). This condition shows explicitly that the boundary conditions at the singularity are correlated with Λ_1 and V_1.

4 Conclusions

It is well established that the stabilization of an extra dimension, by the introduction of a stabilizing potential for the radion field, leads to the resolution of several cosmological paradoxes and to the restoration of the standard Friedmann equation on our brane-universe [15, 16, 17, 18]. The stabilizing potential produces a non-vanishing (55)-component for the energy-momentum tensor which is proportional to the trace of the energy-momentum tensor of our brane. When a "phenomenological" potential for the size of the extra dimension is introduced, this adjustment of T_{55} to the required value is automatic [17].

Here, we have shown that it is possible to stabilize two positive self-energy branes. The ratio of the scale factors on these two branes is determined through the relative detuning of brane self-energies from the limiting value $\sqrt{6|\Lambda_B|/\hat{\kappa}^2}$. The time independence of this solution, equivalent to the cancellation of the effective cosmological constant, comes as an extra condition to which the stabilization mechanism must satisfy. When the brane self-energies are specified, this condition determines uniquely the distance between the two branes.

Next, we considered a candidate mechanism for a dynamical stabilization of the extra dimension. We introduced a bulk scalar field with arbitrary interaction terms on the two

branes. By choosing a vanishing potential for the scalar field in the bulk, the exact solution for the conformal factor, in the presence of the scalar field, was determined for zero, positive and negative bulk cosmological constant. In all cases, the solution for $a(y)$ was accompanied by the appearance of a true spacetime singularity at a finite point y_0 along the extra dimension. The singularity could only be avoided by placing the second brane at a distance $L < y_0$. It was shown that the ratio of the values of the first derivatives, with respect to $\hat{\phi}$, of the interaction terms on the two branes completely determines the ratio of the boundary values of the conformal factor and, moreover, the distance between the two branes. This, according to our analysis, is a generic result independent of the form of the interaction terms of the scalar field on the two branes. However, it would be wrong to conclude that the introduction of a scalar field in the bulk may, indeed, leads to the desired stabilization of the inter-brane distance. In some sense, the fixed size of the extra dimension is the result of *imposing* the time independence of the solution which lead to the overdetermined set of equations. Indeed, we were able to demonstrate that the above result is always accompanied by the need for the correlation of the self-energies of the two branes and the coupling constants of the scalar field with the branes. As in the case of the original Randall-Sundrum model [2], the static solution exists only if the total energy density of one of the two branes (now, given by the sum of the brane self-energy and the scalar interaction term) is negative. This result mars the significance of the successful stabilization of the extra dimension as it introduces an unphysical, and phenomenologically unacceptable, assumption.

It appears that the only solution where the unphysical correlation is not required is the single-brane configuration with the extra dimension ending on the true singularity. The position of this singularity can be interpreted as the size of the extra dimension and depends on the size of the scalar field-brane coupling constant. For a small value of the coupling, this solution generalizes a single brane/infinite dimension configuration discussed in [23] and shows that the presence of a scalar field in the bulk leads to the cutoff of the exponential fall of the scale factor. This static solution could be important as it represents an example, where the effects of presumably large bulk cosmological constant and brane self-energy are completely screened by a massless scalar. Similar observations were made recently in [7]. Unfortunately, the correct way of treating the singularity [25] requires the explicit fixing of boundary conditions for the scalar field and metric at the singular point, which reinstates the fine-tuning problem observed in this work for the two-brane model.

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